JOURNAL OF NATURAL GEOMETRY, 19(2001)73-92. 0963-2654-2001-19-5-73-92.

US$ 20.00. UK£ 10.00.

Measurement of Angles and Elementary Angle Functions Defined by Conformal Cross Ratios

Klaus Ruthenberg

Abstract. Measurement of angles and trigonometric functions are defined with the help of conformal cross ratios, without using any Euclidean structure. Connections between the 2×3 trigonometric functions are the connections between the 2×3 permutations of a real cross ratio.

Introduction

Axiomatic foundations of conformal geometry, independent of Euclidean geometry, was given relatively late. In 1923 *H. Weyl* was apparently unaware of the existence of such foundations [14]. An axiom system was published by *van der Waerden* and *Smid* [13] in the year 1935 and by *Ewald* [1] 1956.

This article is not based on a synthetic axiomatic system but on an invariant theoretical concept of geometry (*F.Klein*, 'Erlanger Programm', [3,4]): Conformal geometry can be characterised by the general group of conformal transformations.

2000 Mathematics Subject Classification: Primary: **30A99; 30C35**; Secondary: **20C99**.

The field $\mathbb{C}$ of normal complex numbers, extended by the element ∞ is used to construct 2-dimensional conformal geometry[1]. Complex numbers are the points of this structure with ∞ as a 'pathological number', the possibility to calculate with this element in $\mathbb{C}$ is very restricted. But the 'point' ∞ is equivalent to every other point for it can be transformed by a conformal mapping into every other point. If we, for illustration only, want to see points z as points of a plane (Gauss, Argand) we should see also these elements as points of a sphere (Riemann, cf. [5,7]). Gauss plane and Riemann sphere are related by a conformal transformation [5,7]. A plane and a sphere are conformally equivalent pictures of $\mathbb{C}$; in conformal geometry a plane is only a sphere in a special location.

Two-dimensional Lobachevskian geometry can be seen as a conformal geometry which possesses an invariant absolute circle (*Poincaré* [9]) [8,7]. I see Euclidean geometry as a conformal geometry which possesses an invariant absolute point. The subgroup of general conformal transformations with a fixed absolute point is isomorphic to *Klein's 'Hauptgruppe'* [4].

Often complex function theory defines (linear) conformal geometry by the group of Möbius transformations $\mathbb{M}$ [5,7] and proves the invariance of the cross ratio with respect to this group. In my eyes conformal geometry is characterised by the invariance of the magnitudes of angles. This article shows that the invariance of real cross ratios is sufficient for the invariance of magnitudes of angles. The invariance of complex cross ratios is sufficient but not necessary for this invariance. Therefore we define conformal geometry not by $\mathbb{M}$ but by a more general group $\mathbb{K} := \mathbb{M} \cup {}^{r}\mathbb{M}^{\lambda}$. Elements of ${}^{r}\mathbb{M}^{1}$ are conformal reflections.

I am interested in a conformal foundation of elementary Euclidean geometry because in this way my triangle model of quaternions [10,11] can be seen as a model situated in a natural 3-dimensional *conformal* space.

[1]Known details of 2-dimensional conformal geometry are collected in H. Schwerdtfegger, *Geometry of Complex Numbers*, New York: Dover, 1979.

1. Conformal reflection and conformal symmetry

Definition 1.1. *In $\mathbb{C}$ a conformal basic transformation $R^1 \in {}^r\mathbb{M}^1$,* **reflection** *or* **symmetry** *is defined by the function*

$$z \mapsto z' : a_{11}z'\bar{z} + a_{12}z' + a_{21}\bar{z} + a_{22} = 0 \qquad (1.1)$$

with

$$|a_{k\lambda}| < 0, a_{k\lambda} = \bar{a}_{\lambda k}$$

and

$$z = \frac{-a_{21}}{a_{11}} \mapsto z' = \infty$$

if $a_{11} \neq 0$;

$$z = \infty \mapsto z' = \infty$$

if $a_{11} = 0$ ($\bar{z}$ is the conjugate of z). The same $A = a_{k\lambda}$ defines $z \mapsto z'$ and $z' \mapsto z$, so we also note $z \leftrightarrow z'$.

It is helpful to see two aspects of this basic transformation: 'Reflection' emphasises that this transformation is a mapping $z \mapsto z'$, with the original z and the image z'. If I regard this transformation as a 'symmetry' I see the set of pairs (z, z') associated by the equation (1.1).

Definition 1.2. *A sequence $z \mapsto z' \mapsto z'' \mapsto z''' \mapsto \dots \mapsto z^{(\lambda)}$ of λ reflections, $\lambda \in \mathbb{N}$, is a* **conformal transformation** *R^λ. If λ is an even number I call this transformation an* **even conformal transformation**. *A transformation constructed by an odd number λ of reflections is an* **odd conformal transformation**. *${}^r\mathbb{M}^\lambda$ is the set of all transformations R^λ.*

Definition 1.3. *An element of $\mathbb{K} = \mathbb{M} \cup {}^r\mathbb{M}^\lambda$ is a* **konformtransformation** *or* **conformal transformation** *K. $\mathbb{K}$ is called the set of* **general conformal transformations**.

Definition 1.4. *The set of fixed points of a conformal reflection (symmetry) is called* **konformkreis** *or* **conformal circle (cocircle)**. *This set is also called* **axis of reflection** *or* **axis of symmetry**.

With $z' = z$ and (1.1) one gets the equation of a konformkreis

$$a_{11}z\bar{z} + a_{12}z + a_{21}\bar{z} + a_{22} = 0, \tag{1.2}$$

also with an hermitian matrix and $|A| < 0$.

Remarks

1. A hermitian matrix A and every matrix $s.A$ with $s \neq 0$ describe the same reflection and the same konformkreis.
2. Because the matrix A describes both the symmetry and its axis we use the same letter A to name this matrix, its symmetry and its axis.
3. We assume $|A| < 0$. If for example $a_{11} = a_{22} = 1, a_{12} = a_{21} = 0$ with $|A| > 0$ we get $z\bar{z} = -1$ as a cocircle equation. But complex numbers with $z\bar{z} = -1$ don't exist.

2. The invariance of conformal circles and symmetries

If $z \neq \infty$, the reflection

$$B : z' \mapsto z : \beta_1 z'\bar{z} + bz' + \bar{b}\bar{z} + \beta_2 = 0 \tag{2.1}$$

can be written

$$z' \mapsto z = -(\bar{b}\bar{z}' + \beta_2)(\beta_1\bar{z}' + b)^{-1} \tag{2.2}$$

If we substitute z of (2.2) into the konformkreis equation (1.2)

$$A : \alpha_1 z\bar{z} + az + \bar{a}\bar{z} + \alpha_2 = 0 \tag{2.3}$$

we get after some straightforward reckoning the transformed equation

$$A' : a_{11}z'\bar{z}' + a_{12}z' + a_{21}\bar{z}' + a_{22} = 0 \tag{2.4}$$

with the matrix elements

$$\begin{aligned} a_{11} &= \alpha_1 b\bar{b} - \beta_1(a\bar{b} + \bar{a}b) + \alpha_2{\beta_1}^2 \\ a_{12} &= b(\alpha_1\beta_2 + \alpha_2\beta_1) - \beta_1\beta_2 a - \bar{a}b^2 \\ a_{21} &= \bar{b}(\alpha_1\beta_2 + \alpha_2\beta_1) - \beta_1\beta_2\bar{a} - a\bar{b}^2 \\ a_{22} &= \alpha_2 b\bar{b} - \beta_2(\bar{a}b + a\bar{b}) + \alpha_1{\beta_2}^2. \end{aligned} \tag{2.5}$$

Again we get an hermitian matrix $a_{k\lambda} = \bar{a}_{k\lambda}$ and it is

$$|a_{\kappa\lambda}| = \begin{vmatrix} \alpha_1 & a \\ \bar{a} & \alpha_2 \end{vmatrix} \cdot \begin{vmatrix} \beta_1 & b \\ \bar{b} & \beta_2 \end{vmatrix}^2. \tag{2.6}$$

Because $|A| < 0$ and $|B| < 0$ it follows with (2.6) that also $|a_{\kappa\lambda}| < 0$. Equation (2.4) has the form of a konformkreis equation, too. The reflection B maps cocircle A into the cocircle A'. Also if we use a sequence of reflections R^λ the image of A is a conformal circle. It is known that a conformal circle is invariant under $\mathbb{M}$. We have thus proved:

Theorem 2.1. *Every conformal transformation $K \in \mathbb{K}$ maps a konforkreis into a konformkreis.*

We also say: A konformkreis is invariant under (with respect to) the general conformal group. Not only the axis of a symmetry but also this symmetry itself is invariant. To show this in ${}^r\mathbb{M}^\lambda$ I map the symmetry

$$A : z \leftrightarrow z^* : \alpha_1 z^*\bar{z} + az^* + \bar{a}\bar{z^*} + \alpha_2 = 0 \tag{2.7}$$

(Compare the equation (2.3) of the cocircle A) by the reflection B of (2.2). Because the equation (2.7) of the symmetry A and the equation of its symmetry axis (2.3) has the same form (we have only to substitute "z" by "z^*"), the same reckoning which leads to (2.4), (2.5), (2.6) also leads to

$$A' : z' \leftrightarrow z^{*\prime} : a_{11} z^{*\prime}\bar{z}' + a_{12} z^{*\prime} + a_{21}\bar{z}^{*\prime} + a_{22} = 0 \tag{2.8}$$

together with (2.5) and (2.6).

Theorem 2.2. *Every conformal transformation $K \in \mathbb{K}$ maps a symmetry into a symmetry.*

Theorem 2.1 and 2.2 say the same thing in different words because hermitian matrix with negative determinant, konformkreis and symmetry are equivalent concepts. Didactical but also scientific considerations make it desirable to state this theorem in two equivalent forms.

Hermitian matrices and symmetries (both in a more general form) have importance in modern physics. Also the usefulness of the konformkreis concept should be discussed in areas where the classical concepts of Euclidean circles, straight lines and lengths have lost significance.

How is my idea of conformal transformations ${}^{r}\mathbb{M}^{\lambda}$ connected with the concept of Möbius transformations $\mathbb{M}$ defined for instance in function theoretical books [5,7]? This connection is described by

Theorem 2.3. *Every conformal transformation R^{λ} with even λ has the 'Möbius form'*

$$z \mapsto z' = (az + b)(cz + d)^{-1} \tag{2.9}$$

with complex a, b, c, d.

Also the set of these even conformal transformations constitute a group. This group of even conformal transformations is a subgroup of the usual Möbius transformations or is identical with this Möbius group. ${}^{r}\mathbb{M}^{\lambda} \subseteq \mathbb{M}$ if λ is even.

Proof. Two reflections produce a product described by an equation of the form (2.9). The product of two transformations (2.9) has again the form (2.9) and this product is equivalent the product of four reflections. An even conformal transformation can be described for every even λ by an equation of the form (2.9). The 'Möbius form' of (2.9) guarantees that also even conformal transformation constitute a group. It is a subgroup of $\mathbb{M}$ for its elements have the form of elements of this group. $\square$

Odd conformal transformations are not elements of $\mathbb{M}$ because odd conformal transformation change the cross ratio (Theorem 5.2) but this cross ratio is invariant in $\mathbb{M}$ [5,7].

Which suppositions secure the identity $\mathbb{M} = {}^{r}\mathbb{M}^{\lambda}$ for even λ? (Which even number λ is sufficient to generate every element of $\mathbb{M}$ as element of ${}^{r}\mathbb{M}^{\lambda}$ for even λ?) This question is interesting, but an answer is not a theme of this article. In $\mathbb{M}$ (and in ${}^{r}\mathbb{M}^{\lambda}$ with even λ) not only conformal circles and circle configurations but also the orientations of a 4-circle are invariant. A transformation with odd λ changes a right orientated 4-circle into a left orientated one (Compare Theorem 5.2). Presumable books of complex analysis and function theory (cf. [5,7]) restrict to Möbius transformations $\mathbb{M}$ because such transformations are analytic functions. Odd transformations are non-analytic ones.

Remarks

1. All conformal circles (symmetries) can especially be standardized by $|A| = -1$. If we use this special standardization the number -1 is invariantly bound with every konformkreis and its conformal basic transformation (cf.(2.6)).
2. The word 'axis' of symmetry remembers that the set of fixed points of a conformal basic transformation appears as an Euclidean straight line if we use the traditional Gauss/Argand plane to illustrate conformal figures and if the point $z = \infty$ is a point of this symmetry axis.
3. The words 'konformkreis' and 'conformal circle' remember that the set of fixed points of a conformal basic transformation appears as an Euclidean circle if we use the traditional Gauss plane to illustrate conformal structures and if the point $z = \infty$ is not a point of this conformal circle.
4. If we use the Riemann sphere of complex analysis [5,7] to illustate conformal structures, all conformal circles (both through and not through $z = \infty$) appear as Euclidean circles.
5. Germans use the word 'kreisgeometrie' ('circle geometry', cf. [1]) to denote the 2-dimensional geometry of conformal circles. Also in the following we often use the old Euclidean word 'circle' to shorten the word 'conformal circle (cocircle)'. But also if we use the same word for Euclidean circles and for conformal circles we have to pay attention to the difference of both conceptions. For example in conformal geometry a circle generally does not

possess a centre; 'radius' of a circle has only in Euclidean but not in conformal geometry an invariant meaning.

6. Only if we restrict the general conformal group to the subgroup of such transformations with an invariant point z_{abs} (I call this point the absolute point), the structure of conformal circles, which do not pass through the absolute point, is identical with the structure of Euclidean circles. For instance in relation to this invariant z_{abs} and its subgroup (which is isomorphic to the Euclidean group of similarities) one can define the centre of an Euclidean circle: The centre z_m of a circle A is the picture of z_{abs} if we map z_{abs} by the reflection A.

7. If we interpret the traditional Gauss plane as an Euclidean plane the point $z = \infty$ is traditionally the absolute point. But it is interesting to use also a point $z \neq \infty$ as absolute point. In this way we get a non-traditional model of the Euclidean structure. This non-traditional picture of Euclidean geometry has some (at least didactical) advantages: For example in this representation Euclidean straight lines can be *seen* as (conformal) circles through the absolute point. From the higher conformal view this non-classical model of Euclidean geometry (with $z_{abs} \neq \infty$) is equivalent to the usual classical model (with $z_{abs} = \infty$). It is only an historically grown convention to identify the 'pathological', not proper number ∞ of the complex field with the 'pathological', not proper point z_{abs} of Euclidean geometry.

I see two aspects of geometry :

1. A transformation K, element of the conformal group $\mathbb{K}$, defines the *variable* aspect of conformal geometry. The elements K bring about the connection between different locations of conformal entities.

2. Only those parts of conformal entities have an objective (invariant, geometrical) meaning which do not change if we change the location of this entity. With help of the axes of symmetries we can construct a set of *invariable* circle entities, a set of conformal basic structures. Every such conformal structure is dual in points and circles, possessing 2^ν points and 2^ν circles ($\nu = 0,1,2,3$). In

the following sections of this article we use 2-circles ($\nu = 1$) to define conformal angles. In a second paper we use 4-circles (ν =2) to define the conformal form of trigonometry. I am especially interested in the conformal geometry of a 4-circle. This circle configuration, seen as an entity of the natural 3-dimensional conformal space, describes the conformal shape of quaternions. And such Hamiltonian complex numbers are seen both as coordinate systems of the 3-dimensional natural space and as elementary geometrical particles in this physical space.

3. Conformal reflections and conformal circles in special locations

A conformal transformation always maps a symmetry (and its axis) into a symmetry (and its axis). Therefore, without loss of generality we can discuss further features of these objects in choosing a 'special location' of such circles and symmetries. (1.2) and (1.1) describe circles and symmetries in 'general location'. A symmetry in special location has a simpler form of its hermitian matrix, for instance

$$A = \begin{pmatrix} 0 & i \\ \bar{i} & 0 \end{pmatrix} \text{or} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.1}$$

with the conformal circle $z = \bar{z}$ or $z\bar{z} = 1$. A little arithmetic proves:

Theorem 3.1. *Every symmetry – in a more general location – can be transformed into the special locations (3.1) so that the axes are $z = \bar{z}$ or $z\bar{z} = 1$.*

4. Two circles in symmetric position

Definition 4.1. *If a reflection A transforms a set of points $\mathbb{F} \subset \mathbb{C}$ into a set $\mathbb{F}'$ in such a way that $\mathbb{F} = \mathbb{F}'$, the circle A is a* **symmetry axis of** $\mathbb{F}$ *; in relation to A the figure $\mathbb{F}$ is in a* **symmetric position**.

If a special location is used it is simple to prove:

Theorem 4.1. *Every conformal circle A possesses infinitely many circles which are in a symmetric position to A.*

If a conformal circle A is the symmetry axis of a circle B, the conformal circle B is also the symmetric axis of A, both circles are in a symmetric position to each other.

5. The cross ratio

A fundamental invariant number of conformal geometry is the cross ratio (cf. [5,7]) w of four points

$$w = (1234) := \frac{z_4 - z_1}{z_4 - z_3} \div \frac{z_2 - z_1}{z_2 - z_3} \tag{5.1}$$

For our purposes we put together known formulae:

$$\begin{aligned} w_1 &:= (1234) = w & w_{-1} &= (1432) = w^{-1} \\ w_2 &:= (1342) = (1-w)^{-1} & w_{-2} &= (1243) = 1 - w \\ w_3 &:= (1423) = (w-1)w^{-1} & w_{-3} &= (1324) = (w-1)^{-1}w \end{aligned} \tag{5.2}$$

$$\begin{aligned} w_1(1-w_3) &= 1 & w_{-3}(1-w_{-1}) &= 1 \\ w_3(1-w_2) &= 1 & w_{-2}(1-w_{-3}) &= 1 \\ w_2(1-w_1) &= 1 & w_{-1}(1-w_{-2}) &= 1 \end{aligned} \tag{5.3}$$

$$w_1 w_2 w_3 = -1 \qquad\qquad w_{-1} w_{-2} w_{-3} = -1 \tag{5.4}$$

I call the three numbers w_κ the *positive* cyclic permutations of a cross ratio; $w_{-\kappa}$ are the *negative* ones. These positive and negative cyclic permutations are related by

$$w_\kappa \cdot w_{-\kappa} = 1, \quad \kappa \;=\; 1, 2, 3 \tag{5.5}$$

and

$$\begin{aligned} w_1 + w_{-2} &= 1 \\ w_2 + w_{-3} &= 1 \\ w_3 + w_{-1} &= 1 \end{aligned} \tag{5.6}$$

A stright forward calculation proves

Theorem 5.1. *If four points z_k define the cross ratio w and if the points z_k are reflected by A,*

$$z_k \longmapsto {z_k}',$$

the cross ratio w' of the image points ${z_k}'$ is the complex conjugate of w:

$$w' = \bar{w}.$$

A transformation R^λ with λ even maps a cross ratio w into the cross ratio $w' = w$ because $w = \bar{\bar{w}}$ for every complex number w.

Theorem 5.2. *Within the group $\mathbb{M}$ and within ${}^r\mathbb{M}^\lambda$ with even λ the cross ratio of four points is invariant. An odd transformation changes such cross ratio into the conjugate number $\bar{w}$.*

If four points are points of the same conformal circle A it follows for the cross ratio of these four points, as fixed points of the reflection A, that

$$w' = w.$$

As consequence of Theorem 5.1 it follows for the reflection A that

$$w' = \bar{w},$$

so that

$$w = \bar{w}.$$

Theorem 5.3. *Four points of the same konformkreis have a real cross ratio. A real cross ratio is invariant in the general conformal group. Also reflections do not change the real cross ratio.*

The position of a point z in relation to two points a, b and a third point z_1 on a cocircle A can be described with help of the real cross ratio of these four circle points

$$w = \frac{b - z_1}{b - z} \div \frac{a - z_1}{a - z}$$

With $z \neq a$ and $z \neq b$ and because w is real we have either $w > 0$ or $w < 0$.

Definition 5.1. *The set of points z on A with $w > 0$ is the* **circle arc** A_1*; the set of points z on A with $w < 0$ is the* **circle arc** A_2.

Theorem 5.4. *Two points a, b on a conformal circle A produce exactly two circle arcs A_1 and A_2.*

For it is either $w > 0$ or $w < 0$.

6. The conformal 2-circle

We want to define 'angles' without using the angle concept of Euclidean geometry. Conformal geometry cannot use this elementary Euclidean angle concept because straight lines and therefore tangents to circles are not defined.

Definition 6.1. *Two points together with two cocircles through these points constitute a* **2-circle**. *The points are the* **corners**, *the circles the* **lines** *of the 2-circle.*

The two corners of a 2-circle separate every line into two arcs (Theorem 5.4).

Definition 6.2.
An **angle** *in a 2-circle is a pair of arcs.*
A **stretched angle** *is an arc pair on the same line.*
Adjacent angles *possess a common arc.*
An angle pair is a pair of **apex angles** *if the two angles produce two stretched angles.*

Remarks

1. For example : A 2-circle AB with the arcs A_1, A_2, B_1, B_2 has the stretched angles A_1A_2 and B_1B_2. Adjacent angles are A_1B_1 and A_2B_1. A_1B_2 and A_2B_1 are a pair of apex angles.

2. The word 'line' of a 2-circle remembers that a conformal 2-circle is corresponding to an Euclidean pair of intersecting straight lines. Generally a

conformal angle possesses 2 arcs and 2 corners (apexes). Only if we illustrate an angle in a Gauss plane in such a way that one corner is the point $z = \infty$ an angle appears in the traditional form, a special 'Euclidean' location. Because in this location $z = \infty$ is the absolute point, an Euclidean angle possesses only one apex, for the second apex is only a 'pathological', not a proper point of the Euclidean plane.

Definition 6.3. *If the two lines of a 2-circle are in a symmetric position to each other these circles are* **right-angled** *or* **orthogonal**.

Definition 6.4. *Two angles have the same* **magnitude** *if they can be conformally mapped into each other.*

7. The characteristic number of a 2-circle

Definition 7.1. *If a circle M is right-angled to both lines of a 2-circle AB this circle M is a* **measuring circle** *of AB.*

Given a 2-circle AB in general location. One can always find a conformal transformation so that the corners z_1 and z_2 of AB get the special location $(z_1, z_2) = (0, \infty)$. Here the lines of the 2-circle are described by the equations

$$A : az + \bar{a}\bar{z} = 0, \qquad B : bz + \bar{b}\bar{z} = 0 \tag{7.1}$$

$a, b \in \mathbb{C}, a \neq b$.
In this location

$$M : z\bar{z} + \alpha = 0 \tag{7.2}$$

(with real $\alpha < 0$) are measuring circles of AB , which follows by reflecting A and B at M. Because every 2-circle can be given the special location (7.1) and because symmetries are invariant with reflections we have proved

Theorem 7.1. *Every 2-circle possesses an infinite number of measuring circles.*

A 2-circle AB (7.1) intersects each of its measuring circles at four points

$$z_{1/3} = \pm\sqrt{\alpha\bar{b}b^{-1}}, \qquad z_{4/2} = \pm\sqrt{\alpha\bar{a}a^{-1}}. \tag{7.3}$$

These four points have the cross ratio

$$w = \left(\frac{\sqrt{\bar{a}a^{-1}} - \sqrt{\bar{b}b^{-1}}}{\sqrt{\bar{a}a^{-1}} + \sqrt{\bar{b}b^{-1}}} \right)^2 \tag{7.4}$$

α does not appear in this formula. Therefore every measuring circle produces the same cross ratio. So the real number w is associated with a 2-circle, independently of the used measuring circle. And because the real w is an invariant cross ratio in the general conformal group, w does not depend on the 2-circle's location.

Definition 7.2. *The real cross ratio w produced by a measuring circle M on a 2-circle AB is the* **characteristic number** *of this 2-circle.*

Equation (7.4) does not show that this characteristic number is a real number. We look for a real parameter ϕ which indicates that $w = \bar{w}$:
If we choose a special representation of B in (7.1) by $b = i = \sqrt{-1}$ and of M in (7.2) by $\alpha = -1$, with $z_{1/3} = \pm 1$ and $z := z_4 = -z_2$ we have

$$w = \left(\frac{1-z}{1+z} \right)^2 = \left(\frac{z^{-1/2} - z^{+1/2}}{z^{-1/2} - z^{+1/2}} \right)^2 = -1 \left(\bar{i} \frac{z^{+1/2} - z^{-1/2}}{z^{+1/2} - z^{-1/2}} \right)^2 \tag{7.5}$$

It is $z = \exp(i\phi)$ because $z\bar{z} = 1$, so that

$$w = -\left(\bar{i} \frac{\exp(i\phi/2) - \exp(-i\phi/2)}{\exp(i\phi/2) + \exp(-i\phi/2)} \right)^2 \tag{7.6}$$

With this formula the real characteristic number w of the 2-circle is described by the real parameter ϕ. For the function

$$\phi/2 \mapsto \bar{i} \frac{\exp(i\phi/2) - \exp(-i\phi/2)}{\exp(i\phi/2) - \exp(-i\phi/2)} \tag{7.7}$$

the symbol $\phi/2 \mapsto \tan(\phi/2)$ is usually used, so that

$$w(\phi) = -\tan^2(\phi/2). \tag{7.8}$$

Because the characteristic number w is a geometrical (invariant) property of the 2-circle, also ϕ has an invariant meaning. We get a bijective correlation between ϕ and w if we restrict to

$$0 < \phi < \pi. \tag{7.9}$$

8. The magnitude of an angle

Given a 2-circle AB and one of its measuring circle M. M may meet the four arcs A_κ of the 2-circle in the four points z_κ ($\kappa = 1, 2, 3, 4$). I mark an angle and its magnitude with the same sign. And for example the angle with the arcs A_1 and A_4 with the sign ϕ_{14}.

Definition 8.1. *The real parameter ϕ according to (7.6) is the magnitude of the angle ϕ_{14}, in symbols $\phi_{14} := \phi$.*

Because this magnitude of an angle is defined with help of the conformal cross ratio we can deduce some elementary angle theorems with help of this cross ratio :

1. $\phi_{14} = \phi_{41}$, because $(1234) = (4321)$. Angles are not orientated.
2. $\phi_{14} = \phi_{32} = \phi_{23}$, because $(1234) = (3412) = (2143)$. Two apex angles have the same magnitude.
3. The magnitudes of adjacent angles possess the sum π, because adjacent angles have inverse cross ratios w_{+1}, w_{-1}, with $w_{+1}w_{-1} = 1$.
4. Orthogonal angles possess the magnitude $\pi/2$, $(1234) = (2341)$, $w_{+1} = w_{-1}$. All angles in a right-angled 2-circle possess the same magnitude $\pi/2$.

We get a special situation if the two corners of a 2-circle are degenerated to a double point. In this situation this double point is a point of the measuring circle too. And two of the four measuring points are identical with this double point. There are essentially two configurations which can be described by $w_1 = 0$, $\phi = 0$ and by $w_1 = \infty$, $\phi = \pi$. If we accept also such degenerated 2-circle we have

$$0 \leq \phi \leq \pi. \tag{8.1}$$

9. Angle functions as permutations of cross ratio

My interpretation of the real cross ratio as the characteristic number of a 2-circle gives a geometrical picture of its 24 permutations.

Section 8 gave the interpretation for the *equalities* of permutations, for example

$$(1234) = (4321) = (3412) = (2143) :$$

An angle and its apex angle have the same magnitude; if we measure an angle and its apex angle the order of its arcs does not matter.
What is the meaning of the 6 *different* permutations $w_{\pm k}$ of (5.2) in section 5? These 6 different permutations of a cross ratio can be described by the same parameter ϕ but in different forms :

$$\begin{array}{lcl@{\qquad\qquad}lcl}
w_{+1} & = & -\left(\bar{i}\frac{e^+-e^-}{e^++e^-}\right)^{+2} & w_{-1} & = & -\left(\bar{i}\frac{e^+-e^-}{e^++e^-}\right)^{-2} \\
w_{+2} & = & +\left(\frac{e^++e^-}{2}\right)^{+2} & w_{-2} & = & +\left(\frac{e^++e^-}{2}\right)^{-2} \\
w_{+3} & = & +\left(\bar{i}\frac{e^+-e^-}{2}\right)^{-2} & w_{-3} & = & +\left(\bar{i}\frac{e^+-e^-}{2}\right)^{+2}
\end{array} \tag{9.1}$$

(with $e^+ := \exp(+i\phi/2), e^- := \exp(-i\phi/2)$) so that

$$\begin{array}{lcl@{\qquad\qquad}lcl}
w_{+1} & = & -\tan^{+2}(\phi/2) & w_{-1} & = & -\tan^{-2}(\phi/2) \\
w_{+2} & = & \cos^{+2}(\phi/2) & w_{-2} & = & \cos^{-2}(\phi/2) \\
w_{+3} & = & \sin^{-2}(\phi/2) & w_{-3} & = & \sin^{+2}(\phi/2).
\end{array} \tag{9.2}$$

Therefore the 2×3 angle functions

$$\begin{array}{lcl@{\qquad\qquad}lcl}
\varphi & \mapsto & \tan\varphi & \varphi & \mapsto & \tan^{-1}\varphi \\
\varphi & \mapsto & \cos\varphi & \varphi & \mapsto & \cos^{-1}\varphi \\
\varphi & \mapsto & \sin^{-1}\varphi & \varphi & \mapsto & \sin\varphi
\end{array} \tag{9.3}$$

(with $\varphi := \phi/2$) and the 2×3 real permutations $w_{\pm k}$ (5.2) reciprocally interpret each other. For instance the inverse relation (5.5) of w_{+k} and w_{-k} can be seen as the inverse relation of the 2×3 angle functions. For example (5.6) takes the form

$$\begin{array}{rcll}
-\tan^2\varphi & + & \cos^{-2}\varphi & = 1 \\
\cos^2\varphi & + & \sin^2\varphi & = 1 \\
\sin^{-2}\varphi & - & \tan^{-2}\varphi & = 1
\end{array} \tag{9.4}$$

The classical equation $\cos^2\varphi + \sin^2\varphi = 1$ appears conformally fitted into a cycle of 3 equations. Without our cross ratio definition of the six elementary angle functions the different signs hide this symmetry.

Two formal things are interesting :

1. The conformal magnitude ϕ of a conformal angle comes together with the classical magnitude of an Euclidean angle. But everywhere in our formulas $\phi/2$ (not ϕ) appears.

2. In each case the cross ratios define the square of an angle function.

Not only the measurement of angles and the definition of angle functions but also a conformal form of Euclidean trigonometry can be founded on the conformal cross ratio.

In another paper I interpret all *non-real* cross ratios as the characteristic numbers of conformal triangles. With help of such a triangle $\tan\varphi$ can be defined by

$$\tan\varphi := v/i \tag{9.5}$$

with an *imaginary* cross ratio v, here v is the characteristic number of a conformal triangle with the angles $\pi/2$, φ, $\pi/2-\varphi$. Between the real number $w = w_{+1}$ of (9.2) and the imaginary number v I found the connection

$$w = v^2 \tag{9.6}$$

with $\varphi = \phi/2$.
This possibility to represent $\tan\phi/2$ with (9.5) as a ratio of two imaginary numbers causes that $\tan\phi/2$ occurs only in a quadratic form, if one restricts to real cross ratios only.

10. Concluding Remarks

Often Minkowski's time transformation $t \mapsto it$ was only understood as a formal trick to get a simpler relativistic formalism. I use imaginary units of lengths and impulses in special relativity [11]. Do these non real units possess a deeper geometrical and physical meaning? In the context of this question it is worthy of note that already a pure and very elementary geometrical concept - the trigonometrical function $\tan\varphi$ - has to be seen as a ratio of two *imaginary* numbers, if a complex number is the characteristic number of a conformal (Euclidean) triangle.

At first the cross ratio was known as a fundamental invariant of *projective* geometry. Formally also the Möbius group $\mathbb{M}$ defines a 1-dimensional projective geometry (with complex points), so the invariance of the cross ratio in $\mathbb{C}$ with respect to $\mathbb{M}$ follows formally in the same way as in real 1-dimensional projective geometry. But on this way the geometrical meaning and importance of this fundamental invariant in 2-dimensional conformal geometry does not come into sight. I could show a real geometrical meaning of the conformal cross ratio by starting with my conformal basic transformations ${}^{r}\mathbb{M}^{\lambda}$, not with the ususal Möbius transformations $\mathbb{M}$. So the measurements of angles and the elementary angle functions could be defined without using Euclidean geometry. I have not found this way of doing things and these results in the mathematical literature. We owe F. Klein the invariant theoretical concept of a geometry [4] but I do not know another author who uses this concept as a methodical principle to develop elementary structures of Euclidean geometry as elements of conformal geometry. My interpretation of Euclidean geometry as geometry of a conformal subgroup with an invariant absolute point and my critical discussion of the role of the point ∞ are necessary steps on this way. The equivalence of konformkreis and hermitian matrix was used to construct a real projective model of 2-dimensional conformal geometry (cf. [2]). A very important element of projective geometry is its dual structure. But in [2] I have not found a hint in which sense 2-dimensional conformal geometry possesses a duality; that basic structures of 2-dimensional conformal geometry, 2^{ν}-circles, are dual in points and circles. Also without success I searched (in publications as [2,6,12]) for my equivalence of hermitian matrix and conformal symmetry, as well as my use of this symmetry to construct the conformal group ${}^{r}\mathbb{M}^{\lambda}$.

I risk a comparison to explain a second aspect of my methodical innovation. Still in historical times human beings perceived our earth as a disk. Today satellites run round the globe. Since Euclid wrote his books, mathematics perceived 2-dimensional Euclidean structures as elements situated in a plane. In my eyes a konformkugel (conformal sphere) is the natural stage of 2-dimensional geometry. If we have this view the absolute Euclidean point loses its special part. This part can be rewritten. We can add such a point as an individual but normal one to classical geometrical structures. The absolute point which has 'chained every Euclidean structure at infinity' can be substituted by individual points if we want to see Euclidean structures as conformal entities. In using this new view an angle possesses not one but two apexes; a triangle possesses not three but four corners. With this methodical innovation old and well-known Euclidean concepts can be seen only as conformal ones.

A conformal interpretation of Euclidean structure leads me to my triangle model of quaternions [10,11]. If we use not the point ∞ but an individual fourth point of these quaternions, every non-real quaternion can be seen as a coordinate system of its konformkugel $\mathbb{C}_i$. The quaternionic i describes the position of this konformkugel in our natural 3-dimensional space. The multiplication of two quaternions A_1, A_2 has to be seen as the interaction of two particles, elements of two conformal spheres with the positions i_1 and i_2. Every conformal sphere can be seen as the stage of a 2-dimensional Euclidean structure, but only 'individual', in relation to an individual point ∞. I compare this with the situation of coordinate systems in special relativity.

References

1. G. Ewald, *Axiomatischer Aufbau der Kreisgeometrie*, Math. Annalen, Bd. 131, p. 353–371 (1956)

2. W. Füchte, *Reelle ebene Möbius-Geometrie*, Dissertation, Freiburg 1982

3. F. Klein, *Elementary Mathematics from an advanced standpoint, Vol. 1, 2.* Dover Publications (Translated from the third German edition, Göttingen 1925)

4. F. Klein, *Vergleichende Betrachtungen über neuere geometrische Forschungen*, Erlangen 1872 (Reprinted in Math. Ann., Bd. 43, p. 63–100)

5. K. Knopp, *Elemente der Funktionentheorie*, Berlin, 1949

6. R. Kulkarni (Editor), *Conformal Geometry*, Vieweg 1988 (A publication of the Max-Planck-Institut f. Math., Bonn)

7. A. I. Maekushevich, *Theory of Complex Variables*, Vol. 1, English edition, Chelsea publishing, 1965

8. H. Meschkowski, *Nichteuklidische Geometrie*, Vieweg, Braunschweig, 1965

9. H. Poincaré, Acta Math., Bd. 1, 1882

10. K. Ruthenberg, *Dreiecke als Elemente algebraischer Körper*, Praxis der Mathematik, Aulis, Cologne, p. 65–70, 1967

11. K. Ruthenberg, *The Quaternionic Structure of 3-dimensional Natural Geometry*, Journal of Natural Geometry, Inan Bhawan, London, Bd. 16, p. 125–140, 1999

12. H. Schwerdtfeger, *Zur Geometrie der Möbius-Transformationen*, Math. Nachrichten, Bd. 18, p. 168–172, 1958

13. B.L. van der Waerden, L. J. Smid, Math. Annalen, Bd. 110, p. 753–776, 1935

14. H. Weyl, *Math. Analyse des Raumproblems*, Berlin, 1923.